

Long paths and cycles in subgraphs of the cube

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Abstract

Let Q_n denote the graph of the n -dimensional cube with vertex set $\{0,1\}^n$ in which two vertices are adjacent if they differ in exactly one coordinate. Suppose G is a subgraph of Q_n with average degree at least d . How long a path can we guarantee to find in G ?

Our aim in this paper is to show that G must contain an exponentially long path. In fact, we show that if G has minimum degree at least d then G must contain a path of length $2^d - 1$. Note that this bound is tight, as shown by a d -dimensional subcube of Q_n . We also obtain the slightly stronger result that G must contain a cycle of length at least 2^d .

1 Introduction

Given a graph G of average degree at least d , a classical result of Dirac [4] guarantees a path of length d in G . Moreover, this bound is best possible as can be seen from K_{d+1} .

Inside the cube Q_n can we improve this bound? That is, given a subgraph G of Q_n with average degree at least d , what is the length of the longest path in G ? The edge isoperimetric inequality for the cube ([1], [5], [6], [7], see [2] for background) says that any subgraph of average degree at least d must have size at least 2^d . In light of this, the above linear bound seems very weak. A natural subgraph of Q_n with average degree at least d is the d -dimensional cube Q_d , the analogue of the complete graph in Q_n , which contains a path of length $2^d - 1$. Must the size of the longest path in G also be exponential?

The main result of this paper answers this question in the affirmative.

Theorem 1.1. *Every subgraph G of Q_n with minimum degree d contains a path of length $2^d - 1$.*

Note that this is best possible as shown by a d -dimensional subcube of Q_n . In fact, the proof of Theorem 1.1 shows that we can always find a longer path in G unless it is isomorphic to Q_d . Using the well known fact that every graph with average degree at least d contains a subgraph with minimum degree at least $\frac{d}{2}$ we obtain the following corollary to Theorem 1.1.

Corollary 1.2. *Every subgraph G of Q_n with average degree at least d contains a path of length at least $2^{\frac{d}{2}} - 1$.*

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We do not know a tight bound for average degree d . We also obtain the corresponding result for the length of the longest cycle in subgraphs of Q_n with large minimum degree.

Theorem 1.3. *Every subgraph G of Q_n with minimum degree d contains a cycle of length at least 2^d .*

In Section 2 we give an overview of the proofs of Theorems 1.1 and 1.3. The theorems themselves are then proved in Sections 3-7.

In Section 8 we show that the lower bound from Theorems 1.1 and 1.3 also extends to subgraphs of the grid graph \mathbb{Z}^n and the discrete torus C_k^n , for all $k \geq 4$. We also give a generalization of Theorem 1.1 and 1.3 to general ‘product-type’ graphs and make some conjectures.

2 Overview

As in the statement of Theorem 1.1, let G be a subgraph of Q_n with $\delta(G) \geq d$. We will view the vertices of Q_n as elements of the power set of $[n]$, $\mathcal{P}[n]$.

A plausible approach to proving Theorem 1.1 is to split G along some direction i to obtain two induced subgraphs G_1 and G_2 consisting of those vertices of G respectively containing and not containing i , for some $i \in [n]$. Provided such a direction is chosen to ensure that $G_1, G_2 \neq \emptyset$, we have $\delta(G_i) \geq d - 1$ and by induction on Theorem 1.1 we have a path of length $2^{d-1} - 1$ in each subgraph. If we could join these two paths into one we would clearly be done. However, as Theorem 1.1 provides no information on where these paths start or end, we can not expect to be able to do this.

This suggests that we strengthen Theorem 1.1 to guarantee an exponentially long path between *any* two vertices x and y of G . In general this is not possible – for example, consider the graph G' obtained by removing all but one edge xy of direction $d + 1$ from the $(d + 1)$ -dimensional cube Q_{d+1} .

However this graph is not 2-connected. The following theorem says that this is the only obstruction to such a strengthening.

Theorem 2.1. *Let G be a 2-connected subgraph of Q_n and a and b be distinct vertices of G . Suppose that $d_G(z) \geq d$ for all $z \in G - \{a, b\}$. Then a and b are joined by a path of length at least $2^d - 2$. Furthermore, unless G is isomorphic to Q_d with a and b at even Hamming distance, G contains an $a - b$ path of length at least $2^d - 1$.*

Note that we do not assume that a or b have degree at least d in Theorem 2.1. This slight weakening of the minimum degree condition will allow us to use induction on various subgraphs of G which would otherwise not be available.

Before continuing with the overview we make a small diversion to introduce some definitions: these are standard (e.g. see [3]).

A subgraph B of a graph G is a *block* of G if B is either a bridge of G or forms a maximal 2-connected subgraph of G . By maximality, $|B_1 \cap B_2| \leq 1$ for any two blocks B_1 and B_2 of G and $G - E(B)$ contains no $x - y$ path between distinct vertices x, y in a block B . Therefore if any two blocks intersect, their common vertex must be a cutvertex and conversely every cutvertex lies in at least two blocks. Since every cycle is 2-connected and an edge is a bridge iff

it does not lie in any cycle, every graph G decomposes uniquely into its blocks B_1, \dots, B_p in the sense that:

$$E(G) = \bigcup_{i=1}^p E(B_i) \text{ and } E(B_i) \cap E(B_j) = \emptyset \text{ if } i \neq j.$$

Suppose now that G is connected. Let $\mathcal{B}(G)$, the *block-cutvertex graph* of G , be the bipartite graph with bipartition $(\mathcal{B}, \mathcal{C})$ where \mathcal{B} is the set of blocks of G , \mathcal{C} is the set of cutvertices of G with Bc an edge if $c \in B$. For a connected graph G , $\mathcal{B}(G)$ is a tree.

The leaves of this tree are all elements of \mathcal{B} and are called *endblocks*. Given an endblock E we will denote its unique cutvertex by $\text{cutv}(E)$. Note that a graph G has only one endblock iff it is 2-connected.

We now return to the overview of the proof of Theorem 2.1.

Lemma 2.2. *Let G be a connected subgraph of Q_n with a and b distinct vertices of G . Then there exists a partition of G into two connected subgraphs G_a and G_b such that $a \in G_a$, $b \in G_b$ and for all $v \in G_c$, $d_{G_c}(v) \geq d_G(v) - 1$, where $c \in \{a, b\}$.*

Proof. Picking a splitting direction i such that a and b differ in coordinate i and forming G_1 and G_2 as before, we have $a \in G_1$ and $b \in G_2$. Let C_b be the connected component of G_2 containing b . Taking G_a to be the connected component of $G - C_b$ containing a and $G_b = G - G_a$ we are done. \square

A central observation in the proof of Theorem 2.1 is that, provided $d \geq 3$, given any endblock E of G_a with $a \notin E$, by induction on Theorem 2.1, E contains a path of length at least $2^{d-1} - 2$ from $\text{cutv}(E)$ to any $y \in E - \text{cutv}(E)$ – here $d \geq 3$ guarantees the E is 2-connected and not a bridge. Since G is 2-connected there must exist $y \in E - \text{cutv}(E)$ with a neighbour in G_b . Thus endblocks like E guarantee ‘endblock paths’ of length at least $2^{d-1} - 1$ from a point in G_a to one in G_b . If we can find a path from a to b containing at least two such endblock paths we would almost be done (we might still be short two or three vertices to give the $2^d - 2$ or $2^d - 1$ bound in G).

For ease of exposition we will prove the following weakening of Theorem 2.1 first. It will allow the reader to focus on the main ideas of the proof of Theorem 2.1 without some distracting details needed to ensure that an $a - b$ path formed from endblock paths is not slightly too short.

Theorem 2.3. *Let G be a 2-connected subgraph of Q_n and $a, b \in V(G)$. Suppose that $d_G(z) \geq d$ for all $z \in V(G) - \{a, b\}$. Then G contains an $a - b$ path of length at least 2^{d-1} .*

Another slight technicality that creeps into the proof of Theorem 2.1 and 2.3 is the possibility that the only partitions of G into G_a and G_b as in Lemma 2.2 above, have a with just one neighbour in G_a or b with just one neighbour in G_b . While all cases can be dealt with simultaneously, we felt for clarity’s sake it is easier to first restrict attention to the case where a partition direction i exists for which both $d_{G_a}(a) \geq 2$ and $d_{G_b}(b) \geq 2$.

Theorem 2.3 is proved in Sections 3-6. Sections 3-5 will focus on the above case, that is, where we can find a partition direction i , such that $d_{G_a}(a) \geq 2$ and

$d_{G_b}(b) \geq 2$. Section 3 will describe the block-cutvertex decomposition structure of G_a and G_b on the absence of an $a - b$ path of length 2^{d-1} formed by joining at least two endblock paths together and Section 4 describes how the endblocks of G_a interact with those of G_b . In Section 5 we show that if G does not contain a path from a to b containing at least two endblock paths then the conditions of Theorem 2.3 hold for a smaller subgraph of G . This allows for an inductive step and completes the proof of Theorem 2.3 in this case.

Section 6 will allow us, using a small modification of the argument from Sections 3-5, to extend from the case $d_{G_a}(a) \geq 2$ and $d_{G_b}(b) \geq 2$ to the general case, therefore proving Theorem 2.3.

Finally in Section 7 we show how to adjust the approach in Sections 3-6 to obtain the optimal bound of Theorem 2.1.

To close this section we note that Theorem 2.1 implies Theorem 1.3.

Proof of Theorem 1.3: Take an endblock E in the block cutvertex decomposition of G . Clearly E is 2-connected and all vertices in $E - \text{cutv}(E)$ have at least d neighbours in E . Pick a neighbour v of $\text{cutv}(E)$ in E . Then by Theorem 2.1 G contains a $\text{cutv}(E) - v$ path P of length at least $2^d - 1$. Combining P with the edge $\text{cutv}(E)v$ we obtain the desired cycle. \square

3 Endblocks in G_a and G_b

To begin we introduce some useful definitions.

Definition 3.1. Let E be an endblock in the block-cutvertex decomposition of G_a (G_b). The *interior* of E is the set $\text{int}(E) = E - \text{cutv}(E)$. A vertex $x \in \text{int}(E)$ is said to be an *exit vertex* of E if x has a neighbour in G_b (G_a). If this neighbour exists, it is unique and is denoted by $p(x)$, x 's *partner*.

Definition 3.2. $\text{Body}(a)$ is the intersection of all blocks of G_a containing a . Let $\text{Core}(a)$ consist of those vertices in $\text{Body}(a)$ that are not cutvertices of G_a .

Definition 3.3. A subgraph K of G_a is said to be a *limb* of a if:

- a is a cutvertex of G_a and $K = G[C \cup \{a\}]$ where C is a connected component of $G_a - a$;
- a is not a cutvertex of G_a and $K = G[C]$ where C is a connected component of $G_a - \text{Core}(a)$.

The *joint* of a limb K , $\text{Joint}(K)$, is the unique vertex $v \in K \cap \text{Body}(E)$.

The reader may find it helpful to examine Figure 1. The circles and ellipses will always denote blocks in the block-cutvertex decomposition of graph.

Proof of Theorem 2.3. The proof is by induction on d where the base case $d = 2$ is trivial. The proof will last until the end of Section 6. Suppose for contradiction, the theorem fails for d and take G to be a minimal counterexample so that Theorem 2.3 holds for all smaller degrees and all graphs G' with $|G'| < |G|$. The following theorem restricts the possibilities for G .

Theorem 2.3'. *Suppose G is a counterexample to Theorem 2.3 so that Theorem 2.3 holds for all smaller degrees and all graphs G' with $|G'| < |G|$. Then there*

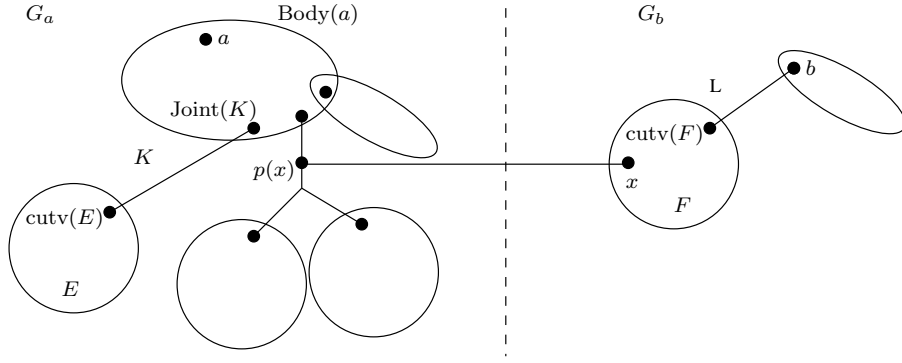


Figure 1: The diagram displays various parts of G_a and G_b . The broken line separates G_a and G_b . In G_a , $\text{Body}(a) \neq \{a\}$ and a has three limbs. In G_b , b is a cutvertex and one of its limbs L contains an endblock F with exit vertex x .

does not exist a direction i such that forming G_a and G_b as in Lemma 2.2, $d_{G_a}(a) \geq 2$ and $d_{G_b}(b) \geq 2$.

Proof of Theorem 2.3. The proof will last until the end of Section 5. Suppose for contradiction that such a G and i exist and form G_a and G_b as above from direction i . Our first lemma describes the block structure of G_a provided we cannot use endblock paths to form an $a - b$ path of length at least 2^{d-1} .

Lemma 3.4. *Given G the following hold:*

- (i) *Every endblock of G_a which does not contain a in its interior must contain at least two exit vertices.*
- (ii) *G_a is not 2-connected.*
- (iii) *a does not lie in the interior of an endblock in G_a .*
- (iv) *a must have at least two limbs.*

Proof. (i) Suppose not and let E be such an endblock. By the 2-connectivity of G , E must contain an exit vertex x . If x were its only exit vertex then every $v \in E - \{\text{cutv}(E), x\}$ has degree least d in $G[E]$ – such v must exist since $d \geq 3$. Then by choice of G , $G[E]$ contains a path P_2 of length at least 2^{d-1} from $\text{cutv}(E)$ to x . Joining a to $\text{cutv}(E)$ in G_a by a path P_1 and $p(x)$ to b in G_b by a path P_3 we have created a path $P_1P_2P_3$ of length 2^{d-1} from a to b , a contradiction.

(ii) Suppose G_a is 2-connected. First consider the case where G_b is not 2-connected. Let E be an endblock in G_b not containing b in its interior and take x to be an exit vertex of E with $p(x) \neq a$ – this exists by (i). Then by induction on d , there are paths P_1 in G_a from a to $p(x)$ and P_2 in $G[E]$ from x to $\text{cutv}(E)$ both of length at least 2^{d-2} . Taking a path P_3 from $\text{cutv}(E)$ to b in G_b we have constructed a path $P = P_1p(x)xP_2P_3$ from a to b of length at least 2^{d-1} , a contradiction.

If G_b is 2-connected then the same proof as in (i) shows that G_b must contain two exit vertices one of which x has $x \neq b$ and $p(x) \neq a$. By induction on d we obtain endblock paths from a to $p(x)$ in G_a and from x to b in G_b both of

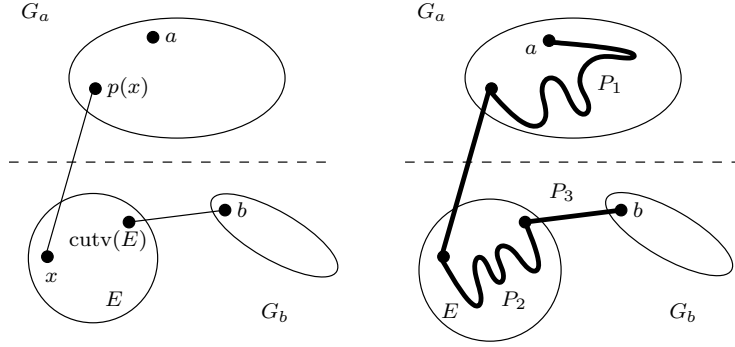


Figure 2: Path P constructed in Lemma 3.4(ii). Curved paths like P_1 and P_2 will represent endblock paths of length at least 2^{d-2} throughout.

length at least 2^{d-2} . Joining the two with edge $xp(x)$, G contains the required path, a contradiction.

(iii) Suppose a did lie in the interior of an endblock E of G_a . As $d_{G_a}(a) \geq 2$ E is 2-connected, so by induction on d we have an endblock path P_1 from a to $\text{cutv}(E)$ in $G[E]$ of length at least 2^{d-2} . Now by (ii) G_a is not 2-connected and so it contains a second endblock E' , with an exit vertex x . Again by induction on d , $G[E']$ contains an endblock path P_3 from $\text{cutv}(E')$ to x of length 2^{d-2} . Join $\text{cutv}(E)$ to $\text{cutv}(E')$ by a path P_2 in G_a and $p(x)$ to b by a path P_4 in G_b . Combining all of these paths we have a path $P_1P_2P_3xp(x)P_4$ from a to b of length at least 2^{d-1} , a contradiction.

(iv) This follows from (ii) and (iii) as if G_a is not 2-connected and a does not lie in the interior of any endblock, a must have at least two limbs. \square

Note that by symmetry of a and b , Lemma 3.4 also applies on replacing a with b . The next proposition dispenses with the simplest case we can use endblock paths to build our path of length 2^{d-1} from a to b .

Proposition 3.5. *For an exit vertex x in an endblock E of G_a , $p(x)$ can never lie in the interior of an endblock F of G_b .*

Proof. From Lemma 3.4(iii) $a \notin \text{int}(E)$ and $b \notin \text{int}(F)$. Pick a path P_1 in G_a from a to $\text{cutv}(E)$ and a path P_4 in G_b from $\text{cutv}(F)$ to b . Since E is 2-connected and all $v \in E - \{\text{cutv}(E), x\}$ have degree at least $d-1$ in $G[E]$, by induction $G[E]$ contains a path P_2 of length at least 2^{d-2} from $\text{cutv}(E)$ to x . Similarly $G[F]$ contains a path P_3 of length at least 2^{d-2} from $p(x)$ to $\text{cutv}(F)$. Combining these gives an $a-b$ path $P = P_1P_2xp(x)P_3P_4$ of length at least 2^{d-1} , a contradiction. \square

4 The Interaction Digraph

Let K_1, \dots, K_r be the limbs of a and L_1, \dots, L_s be the limbs of b . Note that by Lemma 3.4(iv) $r, s \geq 2$.

We form an auxiliary bipartite multidigraph $H = (A, B, \vec{E})$ which will represent the interaction between the limbs and cores of a and b . Let $A = \{K_1, \dots, K_r\}$ and $B = \{L_1, \dots, L_s\}$. Additionally, adjoin $\text{Core}(a)$ to A and

$\text{Core}(b)$ to B if they are non-empty. Given an endblock E of G_a there exists an exit vertex x with $x \neq a$ and $p(x) \neq b$ by Lemma 3.4(i) and (iii). Pick exactly one such exit vertex x_E for each such endblock E and adjoin a directed edge to H from K to $W \in B$ where E is contained in limb K and $p(x_E) \in W$. Similarly, for each endblock F in L we pick an exit vertex $y_F \in F$ with $p(y_F) \neq a$ and add a directed edge to H from L to V where $p(y_F) \in V$.

Note that by Proposition 3.5 we never choose an exit vertex x_E for some E and y_F for some F such that $p(x_E) = y_F$. Also since any limb of a or b contains an endblock, every limb vertex in H must have outdegree at least one and core vertices have no outneighbours.

We shall study the component structure of H . The next two lemmas say that this must be very restricted. Together they will allow us to find a connected component C of H consisting entirely of limbs. The inductive step in Section 5 will take place on the subgraph of G corresponding to this C .

Lemma 4.1. *H cannot contain an undirected path of length three.*

Proof. Suppose we have such an undirected path $Q = V_0V_1V_2V_3$ in H and assume $V_0 \in A$. Each directed edge \overrightarrow{VW} of Q gives an endblock in V with exit vertex x , such that $p(x) \neq b$ and $p(x) \in W$. These endblocks are distinct by the construction of H and in each we can find an endblock path of length 2^{d-2} from its cutvertex to this exit vertex by induction on d . We claim that we can form an $a - b$ path P which extends all three of these paths. As such a path has length at least $3(2^{d-2}) > 2^{d-1}$, this contradicts our choice of G and proves the lemma.

We will construct our path by forming paths P_i in each V_i and eventually join them into one. The start point of P_i will be denoted by a_i and its end point by b_i . We first choose these vertices.

If $\overrightarrow{V_iV_{i+1}}$ is an edge of Q there is an endblock E in V_i with an exit vertex x such that $p(x) \in V_{i+1}$. In this case let $b_i = x$ and $a_{i+1} = p(x)$. If $\overleftarrow{V_iV_{i+1}}$ is an edge of Q this gives an endblock E in V_{i+1} with an exit vertex x such that $p(x) \in V_i$. In this case let $b_i = p(x)$ and $a_{i+1} = x$. We set

$$a_0 = \begin{cases} \text{Joint}(V_0) & \text{if } V_0 \text{ is a limb of } a; \\ b_0 & \text{if } V_0 = \text{Core}(a) \end{cases}$$

$$b_3 = \begin{cases} \text{Joint}(V_3) & \text{if } V_3 \text{ is a limb of } b; \\ a_3 & \text{if } V_3 = \text{Core}(b). \end{cases}$$

Note that b_i and a_{i+1} are adjacent for $i \in \{0, 1, 2\}$ and $a, b \notin \{b_0, a_1, b_1, a_2, b_2, a_3\}$.

We now build the paths P_i from a_i to b_i in each V_i , where V_i is a limb. We claim we can choose P_i so that neither a nor b are interior vertices of P_i (that is, they can lie on P_i , but only as end vertices) and P_i has length at least 2^{d-2} if V_i has one outneighbour on Q and 2^{d-1} if V_i has two. Indeed, if V_i has one outneighbour in Q then one of a_i or b_i must be an exit vertex of an endblock E of V_i . Without loss of generality this is a_i . Then $b_i \notin \text{int}(E)$ by Proposition 3.5 and $G[E]$ contains a path of length 2^{d-2} from a_i to the cutv(E). Since $V_i - \{a, b\}$ is connected for all i from the definition of a limb, we can extend this path from cutv(E) to b_i as required. The case where V_i has two outneighbours on Q is identical, using the same argument in two endblocks of V_i and joining their cutvertices in V_i .

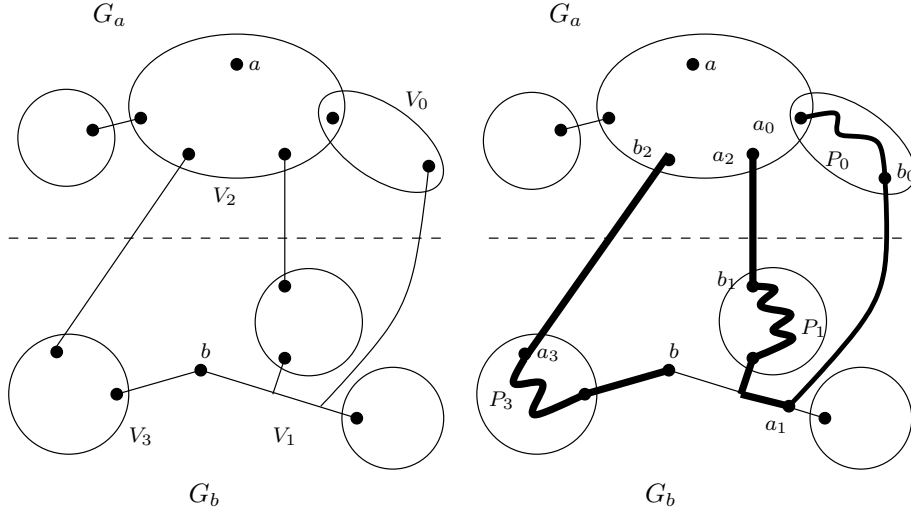


Figure 3: An illustration of Lemma 4.1 in the case where $V_2 = \text{Core}(a)$ and V_0V_1 , V_1V_2 and V_3V_2 are directed edges of Q . As in the proof of Lemma 4.1, 2-connectivity can be used in $\text{Body}(a)$ to find vertex disjoint paths from $\{a_0, a_2\}$ to $\{b, b_2\}$.

Finally we combine the P_i paths. We first deal with the case where neither $\text{Core}(a)$ nor $\text{Core}(b)$ occur as interior vertices of Q . Combining the paths above we have an $a_0 - b_3$ path $P' = P_0b_0a_1P_1b_1a_2P_2b_2a_3P_3$. If $\text{Body}(a) = \{a\}$ then P' starts at a so we only need to extend P' to start at a when $\text{Body}(a) \neq \{a\}$. In P' as constructed above, $\text{Body}(a) \cap P'$ contains a_0 and at most one other vertex. Since $\text{Body}(a)$ is 2-connected it contains a path P'_1 from a to a_0 avoiding this vertex. Finding a similar path P'_2 from b_3 to b in $\text{Body}(b)$ if $\text{Body}(b) \neq \{b\}$ we may take $P = P'_1P'P'_2$.

If Q contains one of the Core vertices, without loss of generality let it be $\text{Core}(a)$. If $\text{Core}(a)$ occurs as an interior vertex of Q , it must be V_2 . $\text{Body}(a)$ then contains distinct a_0, a_2, b_2 and we have two paths $P'_1 = P_0b_0a_1P_1b_1a_2$ from a_0 to a_2 and $P'_2 = b_2a_3P_3$ from b_2 to b_3 as in Figure 3. From the choice of the a_2 and b_2 above and the fact that a is not a cutvertex we have $a \notin \{a_0, a_2, b_2\}$. Then by 2-connectivity $\text{Body}(a)$ contains two vertex disjoint paths from $\{a_0, a_2\}$ to $\{a, b_2\}$. Piecing these paths together with P'_1 and P'_2 we obtain an ab_3 -path P' . If $\text{Body}(b) = \{b\}$ we are done since $b = b_3$. Otherwise we extend P' using 2-connectivity as above to find an $a - b$ path of length at least 2^{d-1} , a contradiction to the choice of G . \square

Note that Lemma 4.1 guarantees that H has at least two connected components. The next lemma further limits H . Its proof is very similar to that of Lemma 4.1.

Lemma 4.2. *Suppose that we have $\text{Body}(a) \neq \{a\}$. Then no component of H contains two vertices of A .*

Proof. Suppose H has such a component C . Then by Lemma 4.1, C consists of vertices V_1, \dots, V_t in A and vertex W in B . At most one of V_1, \dots, V_t, W can be a core vertex as there is no edge between $\text{Core}(a)$ and $\text{Core}(b)$ in H .

If $W = \text{Core}(b)$ then V_1 and V_2 must be limbs and these guarantee two vertex disjoint paths P_1, P_2 from vertices $a_1, a_2 \in \text{Body}(a)$ to vertices $b_1, b_2 \in \text{Body}(b)$ both of length at least 2^{d-2} . By Lemma 4.1 and Lemma 3.4(iv) H must contain a second component C' containing a limb of b which guarantees the existence of a third path P_3 from a vertex $a_3 \in \text{Body}(a)$ to $b_3 \in \text{Body}(b)$ of length 2^{d-2} . Using identical 2-connectivity arguments in both $\text{Body}(a)$ and $\text{Body}(b)$ as in Lemma 4.1 we can combine these three paths into one from a to b , a contradiction.

If $W \neq \text{Core}(b)$ then C guarantees a path P_1 of length 2^{d-1} between two vertices a_1 and a_2 in $\text{Body}(a)$ with $b \notin P_1 \cap \text{Body}(b)$ and $|P_1 \cap \text{Body}(b)| \leq 1$. Again from a second connected component of H we obtain a disjoint path P_2 from an element $a_3 \in \text{Body}(a)$ to $b_1 \in \text{Body}(b)$. Once more, with an application of 2-connectivity in $\text{Body}(a)$ and a possible application in $\text{Body}(b)$ we find an $a - b$ path extending both P_1 and P_2 , a contradiction. \square

Again the same applies switching a with b . As mentioned before Lemma 4.1 the previous two lemmas imply that H contains a connected component C consisting entirely of limbs. If not, each component of H would contain one of $\text{Core}(a)$ or $\text{Core}(b)$ and thus H would contain at most two connected components. Since H contains no path of length 3 by Lemma 4.1, it must have exactly two components, one containing $\text{Core}(a)$ and the other containing $\text{Core}(b)$. But as A contains $\text{Core}(a)$ and at least two limbs, two of these must lie in the same connected component contradicting Lemma 4.2.

5 The Inductive Step

Let C be the component of H consisting entirely of limbs of a and b guaranteed from Section 4 and write G_C for the subgraph $G[\bigcup_{W \in C} V(W)]$ of G . Notice that G_C must contain exactly one vertex a_C in $\text{Body}(a)$ and one vertex b_C in $\text{Body}(b)$ - if $\text{Body}(a) = \{a\}$ then $a_C = a$, if not $C \cap A = \{V\}$ by Lemma 4.2 and $a_C = \text{Joint}(V)$.

Our final lemma before we complete the proof of Theorem 2.3' allows us to find a subgraph of G_C which will either also satisfy the conditions of Theorem 2.3 or build at least half of the path we are looking for from any edge entering it. Before stating it we give one last definition.

Definition 5.1. Given a graph G and $S \subset V(G)$ define the $\text{span}(S)$ in G to be the subset of $V(G)$ consisting of all vertices which lie on a path between two elements of S .

Note that we include paths of length zero in this definition, so that $S \subset \text{span}(S)$.

Lemma 5.2. *Let C and G_C be as above. Then G_C has a 2-connected subgraph J containing two vertices $a' \in G_a$ and $b' \in G_b$ with the following properties:*

- (i) *every vertex $v \in J - \{a', b'\}$ has degree at least $d - 1$ in J and all the neighbours of v in G_C are contained in J .*
- (ii) *for any vertex $v \in J - \{a', b'\}$, J contains an $a' - v$ path not containing b and a $b' - v$ path not containing a , both of length at least 2^{d-2} .*

Proof. Suppose first that C consists of two vertices with a limb K in $A \cap C$ and L in $B \cap C$. Let S be the subset of vertices of $K - a$ with a neighbour in $L - b$ and T the subset of $L - b$ with a neighbour in $K - a$. Note that by definition of C each endblock in K has an exit vertex x with $p(x) \in L - a$ and by Proposition 3.5 $p(x)$ cannot lie in the interior of an endblock of L . As the same holds true for the endblocks in L , both $|S|, |T| \geq 2$.

Consider the graph $G' = G[\text{span}(S) \cup \text{span}(T)]$. This is 2-connected and contains all vertices of the endblocks of K in G_a and L in G_b . Furthermore, G' restricted to G_a is a union of blocks of G_a . Such a subgraph must be joined to the rest of the G_a by a single cutvertex a'_0 . The same holds true for G' restricted to G_b with cutvertex b'_0 . While this is almost the graph we will take as J , a and b may be vertices of K and L respectively, not lie in G' but still have a neighbour in $G' - \{a'_0, b'_0\}$ which would not allow (i) to hold. If this occurs with a say, then adjoin it to S and run the same construction as above with this new S to obtain a new graph G'' . Let the a'_1 be the cutvertex with the rest of G_a and b'_1 the cutvertex with the rest of G_b . The same problem may now occur with G'' and b . If so, adjoining b to T we obtain our final graph G''' , which again is 2-connected with cutvertex a'_2 with the rest of G_a and cutvertex b'_2 with the rest of G_b . It should be clear that (i) now certainly holds for this graph and we may take $J = G'''$, $a' = a'_2$ and $b' = b'_2$.

To see that J contains an $a' - v$ path as claimed in (ii) we first consider the case with $b' \neq b$. Now J contains an endblock E from G_a and an endblock F from G_b and as v is in at most one of these, assume $v \notin F$. Taking \tilde{J} to be the graph formed from J by contracting $\text{int}(F)$ to a single point f it is easily seen \tilde{J} is still 2-connected. As neither a' nor b' can lie in the interior of F , by 2-connectivity \tilde{J} contains two vertex disjoint paths from the set $\{a', v\}$ to $\{\text{cutv}(F), f\}$. These paths give two paths in J , P_1 from a' to say $w \in \text{int}(F)$ and P_3 from v to $\text{cutv}(F)$. By induction on Theorem 2.3 $G[F]$ contains a path P_2 of length 2^{d-2} from $\text{cutv}(F)$ to w . $P = P_1 P_2 P_3$ now works for the $a' - v$ path claimed in (ii).

If $b' = b$, using the same argument as above we might use b in one of the paths P_1 or P_3 . However in this case we have the following:

$J - b$ has a 2-connected subgraph J' containing all of $J \cap G_a$ and an endblock of G_b .

To see this, look at the block-cutvertex decomposition of $J - b$. This contains exactly one block J' with vertices in both G_a and G_b and this block contains all of $J \cap G_a$ by construction. Since any endblock of G_b contained in J has an exit vertex whose partner lies in $J \cap G_a$, all such endblocks lie in J' . Given v as above, choose a minimal path P_0 not containing b from v to some $v' \in J'$. We may now run the same argument as above in J' to obtain a $a' - v'$ path P' of length at least 2^{d-2} . Combining the resulting path with P_0 we are done. By symmetry this finishes the case where C consists of two vertices.

We now deal with the case where C consists of more than two vertices. By Lemma 4.1 and Lemma 4.2 we may assume C consists of a limb K of a , limbs L_1, \dots, L_t of b and that $\text{Body}(b) = \{b\}$. Let $L_1, \dots, L_{t'}, t' \leq t$, be the limbs containing at least two vertices other than b with neighbours in $K - a$. Note that $t' \geq 1$ since some L_i must be an outneighbour of K in H . We will first work with these limbs and add in the rest if needed later.

As above, for each $i \in [1, t']$ let $S_i \subset V(K)$ consist of those vertices in $K - a$ with a neighbour in $L_i - b$ and $T_i \subset V(L_i)$ consist of the vertices of $L_i - b$ with a neighbour in $K - a$. Taking $G_i = G[\text{span}(S_i) \cup \text{span}(T_i)]$ this graph is 2-connected for all $i \in [1, t']$. Note that $G_i \cap G_b$ must again have exactly one cutvertex with the rest of G_b , say c_i . Once more there are concerns that $a \in K$ or $b \in L_i$ are not contained in G_i but have neighbours in it as this would violate (i) above. We will again adjoin them to S_i and T_i as needed. While the condition to adjoin a to S_i is exactly the same as above, that is adjoin a to S_i if a has a neighbour in $G_i \cap G_b - c_i$, the condition to adjoin b to T_i needs a slight variation since $G_i \cap G_a$ may have more than one cutvertex separating it from the rest of G_a . With a little foresight, take $c'_i \in G_i \cap G_a$ to be the unique cutvertex of G_a separating $G_i \cap G_a$ from a (possibly a itself) and adjoin b to T_i if b has a neighbour in $G_i \cap G_a - c'_i$. For each $i \in [1, t']$, having added the vertices to S_i and T_i as demanded by these conditions, let S_i and T_i now refer to the new sets and G_i to the new graphs formed from them.

For all $M \subset [1, t]$ define $S_M = \bigcup_{j \in M} S_j$. Beginning with an initial list $\{S_1, \dots, S_{t'}\}$ repeatedly remove S_M and S_N from the list and adjoin $S_{M \cup N}$ if $|\text{span}(S_M) \cap \text{span}(S_N)| \geq 2$. This procedure eventually terminates, and in the final list there must exist some S_M for which $G[\text{span}(S_M)]$ has a single cutvertex separating it from the rest of G_a . To see this note that for each I , $G[\text{span}(S_I)]$ is a connected union of blocks of G_a and so corresponds to a subtree of the block graph $\mathcal{B}(G_a)$. Since each leaf of $\mathcal{B}(G_a)$ is contained in some $G[\text{span}(S_I)]$ and no two $G[\text{span}(S_I)]$ and $G[\text{span}(S_{I'})]$ share a block, one of these, $G[\text{span}(S_M)]$ say, must have a single cutvertex a' with the rest of G_a .

Now we deal with the limbs L_j where $j \in [t' + 1, t]$. Form a set $N \subset [t' + 1, t]$ with $i \in N$ if some $v \in L_i - b$ has a neighbour in $\text{span}(S_M) - a'$ and let $I = M \cup N$.

If $I = \{i\}$ for some i then $N = \emptyset$ and we can run the same argument as in the case where C consists of two vertices replacing S with S_i and T with T_i . If not, take $J = G[\bigcup_{i \in I} V(L_i) \cup \text{span}(S_I)]$. J is 2-connected and taking $b' = b$, J satisfies (i).

Finally we show that (ii) holds for J . Given $v \in J - \{a', b'\}$ suppose we are looking for an $a' - v$ path avoiding b of length at least 2^{d-2} . We again claim:

$J - b$ has a 2-connected subgraph containing all of $J \cap G_a$ and an endblock F of G_b .

It suffices to show that $G[\bigcup_{i \in M} V(L_i) \cup \text{span}(S_M)]$ contains such a subgraph. Exactly as in the case where C consists of two limbs, each $G[\text{span}(S_i) \cup \text{span}(T_i)] - b$, $i \in [1, t']$, has a 2-connected subgraph G'_i containing all of $\text{span}(S_i)$. Moreover one of the subgraphs $G[\text{span}(T_i)] - b$ must contain an entire endblock F (since $t' \geq 1$). Now the subgraph $G[\bigcup_{i \in M} V(G'_i)]$ of $J - b$ is 2-connected by construction of M , contains all vertices of $J \cap G_a$ and endblock F in G_b , proving the claim. Using this subgraph, exactly as in the case where C consists of two limbs, we obtain an $a' - v$ path of desired length.

An identical argument gives the $b' - v$ path as claimed in (ii). □

Now we are ready to finish the proof of Theorem 2.3'. Let J be the subgraph of G guaranteed from Lemma 5.2. If there is no edge between $J - \{a', b'\}$ and $G - J$, all $v \in J - \{a', b'\}$ have degree at least d in J and by choice of G , J

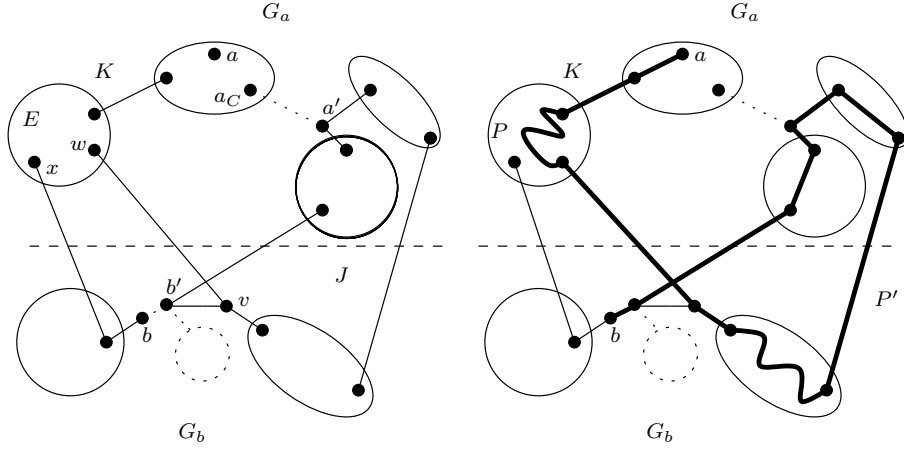


Figure 4: An illustration of the case $w \in \text{int}(E)$ in the proof of Theorem 2.3'. The broken dotted pieces represent vertices in G_C that are left out of J .

contains an $a' - b'$ path P of length at least 2^{d-1} . Extending this path from a' to a and b' to b , G contains an $a - b$ path of length 2^{d-1} , a contradiction.

So such an edge must exist, joining say $v \in J - \{a', b'\}$ to $w \notin G_C$ by Lemma 5.2(i). We may assume $w \in G_a$. First suppose $w \in \text{Body}(a)$ – this is only possible if $\text{Body}(a) \neq \{a\}$, by Lemma 5.2(i). Applying Lemma 5.2(ii) we have an $a' - v$ path in J not containing b of length at least 2^{d-2} . This path extends in C to an $a_C - v$ path P_1 , where again $a_C = G_C \cap \text{Body}(a)$. Now taking a limb K of a not in C , K guarantees a path P_2 from $\text{Joint}(K)$ to b of length at least 2^{d-2} disjoint from P_1 . Thus we have a path $P_1 vw$ joining two vertices a_C and w in $\text{Body}(a)$ and a path P_2 from $\text{Joint}(K)$ to b . By 2-connectivity, $\text{Body}(a)$ contains two vertex disjoint paths P_3 and P_4 from $\{a, \text{Joint}(K)\}$ to $\{a_C, w\}$. Combining all four of these paths we obtain an $a - b$ path P extending both P_1 and P_2 . But this path has length at least 2^{d-1} , contradicting our choice of G .

So we may assume $w \notin \text{Body}(a)$. Then $w \in K$ for some limb K of a , where by Lemma 5.2(i) $K \notin C$. Let E be an endblock of K and take x_E to be the exit vertex of E chosen in the construction of H in Section 4. Note that $p(x_E) \notin G_C$, since $K \notin C$. Depending on whether or not w lies in $\text{int}(E)$ we can construct a path P from w to either $\text{Joint}(K)$ or to x of length 2^{d-2} by induction on d in Theorem 2.3. If P is from w to $\text{Joint}(K)$ adjoin it via the edge vw to the $v - b'$ path in J guaranteed by Lemma 5.2(ii) (see Figure 4). This gives a $\text{Joint}(K) - b'$ path P' of length at least 2^{d-1} which extends to an $a - b$ path. If P is from w to x_E then we combine this path with the $v - a'$ path in J guaranteed by Lemma 5.2(ii). Again this gives a $a' - p(x_E)$ path of length at least 2^{d-1} which extends to an $a - b$ path. This proves Theorem 2.3'. \square

6 Removing the Degree Assumption

Recall in Lemma 2.2, we obtained a splitting of G into two pieces, one containing a and the other containing b by choosing a direction i on which a and b differ when viewed as elements of $\mathcal{P}[n]$. In general it is not possible to choose such

a direction to ensure that both $d_{G_a}(a) \geq 2$ and $d_{G_b}(b) \geq 2$ – for example, a and b could be adjacent with both having only one other neighbour in G . It is however always possible to choose such a direction i if $d_G(a) \geq 3$ and $d_G(b) \geq 3$. We will therefore assume that $d_G(a) = 2$.

The condition $d_{G_a}(a) \geq 2$ and $d_{G_b}(a) \geq 2$ as used above allowed us to ensure that all endblocks of G_a and G_b contain long paths, which is clearly false if a has a single neighbour a' in G_a . This in turn guaranteed that a had at least two limbs in G_a which was crucially used numerous times in our analysis of H e.g. Lemma 4.2. In this section, we extend the arguments of Theorem 2.3' to prove Theorem 2.3.

Lemma 6.1. *The following hold:*

- (i) $G - a$ is a 2-connected graph.
- (ii) We can choose a splitting direction i such that after forming G_a and G_b as above from this i , $d_{G_a}(a) \geq 2$ or $d_{G_b}(b) \geq 2$

Proof. (i) If $G - a$ is not 2-connected it has at least two endblocks in its block-cutvertex decomposition, one of which E has $b \notin \text{int}(E)$. Now since G is 2-connected, a must be joined to the interior of all the endblocks of $G - a$. As $d_G(a) = 2$, $G - a$ has exactly two endblocks with a having exactly one neighbour in the interior of each. Let w be this neighbour in E .

Now E is 2-connected (as $d \geq 3$) and all vertices in $G[E] - \{\text{cutv}(E), w\}$ have degree at least d in $G[E]$. Since G is the smallest counterexample and $G[E]$ is a non-spanning subgraph of G , it contains a path P of length 2^{d-1} from w to $\text{cutv}(E)$. Extending this path on either side to a and b respectively, we have an $a - b$ path of length at least 2^{d-1} , a contradiction.

(ii) We can always choose such a direction if a and b are at Hamming distance at least three in Q_n or if one of a or b have degree greater than 2 in G . So a and b must be at Hamming distance one or two in G and both have degree exactly two.

First consider a and b at Hamming distance one. If they are not adjacent in G we may choose the direction on which they differ for i so we can assume they are adjacent. Then a and b both have one other neighbour in G , a' and b' respectively. Now if $G - \{a, b\}$ is 2-connected we can apply Theorem 2.3 to $G - \{a, b\}$ with a' and b' replacing a and b to obtain an $a' - b'$ path of length at least 2^{d-1} . Adjoining the edges aa' and bb' to this path we have an $a - b$ path of length $2^{d-1} + 2$, more than enough. If $G - \{a, b\}$ is not 2-connected it is easily seen that a' and b' must lie in the interior of different endblocks of $G - \{a, b\}$. We can therefore find a path from a' to b' in $G - \{a, b\}$ which extends two endblock paths. Adjoining the edges aa' and bb' to this path again we have an $a - b$ path of length at least $2^d + 2$.

If a and b are at Hamming distance two in the cube, we can always find such a direction i unless a and b are joined to the same two neighbours in G , a' and b' say. Then $\{a, a', b, b'\}$ form a C_4 with a opposite b . Working with $G - \{a, b\}$, a' and b' as above, we again obtain an $a - b$ path of desired length in G . \square

From Lemma 6.1(ii) we can now assume that we have chosen a partition direction i such that $\deg_{G_b}(b) \geq 2$. Lemma 3.4(i)-(iv) still hold for G_b with the same proofs as above. In particular b has at least two limbs. In order for our main argument in Theorem 2.3' to be inapplicable, a must have exactly one

neighbour a' in G_a and one neighbour v in G_b . We may also assume that $v \neq b$ as otherwise by Lemma 6.1(i) we could apply Theorem 2.3 to $G - a$ taking a' in place of a .

Lemma 6.2. *We have $v \in \text{int}(E_v)$ for some endblock E_v of G_b .*

Proof. Suppose not. From Lemma 3.4(iii) b does not lie in the interior of an endblock of G_b and by Lemma 3.4(iv) G_b contains two vertex disjoint paths P_1 from v to $\text{cutv}(E_1)$ and P_5 from $\text{cutv}(E_2)$ to b , where E_1 and E_2 are two endblocks of G_b . Taking exit vertices x_1 and x_2 of E_1 and E_2 respectively, by induction on d , $G[E_1]$ contains a path P_2 of length at least 2^{d-2} from $\text{cutv}(E_1)$ to x_1 and $G[E_2]$ contains a path P_4 of length at least 2^{d-2} from x_2 to $\text{cutv}(E_2)$. Taking a path P_3 from $p(x_1)$ to $p(x_2)$ in $G_a - a$ and combining the paths, G contains an $a - b$ path $avP_1P_2x_1p(x_1)P_3p(x_2)x_2P_4P_5$ of length at least 2^{d-1} , a contradiction. \square

We again construct an interaction digraph H but this time it is built from the limbs of a' and b instead of those of a and b . Note that $\{a, a'\}$ is a limb of a' and so both a' and b have at least two limbs. Take $H = (A', B, \vec{E})$ to be a bipartite multidigraph on vertex sets $A' = \{K_1, \dots, K_r\}$ and $B = \{L_1, \dots, L_s\}$, the set of limbs of a' and b respectively. We also adjoin $\text{Core}(b)$ to B if it is non-empty ($\text{Core}(a') = \emptyset$ since a' is a cutvertex of G_a). Now each endblock of G_a or other than $\{a, a'\}$ contains at least two exit vertices, as in Lemma 3.4(i). Therefore for each endblock E of G_a or G_b other than $\{a, a'\}$ we can pick an exit vertex x_E with $p(x_E) \neq a', b$. From Lemma 6.2 we can pick $x_{E_v} = v$. Now adjoin a directed edge from $K \in A'$ to $L \in B$ for each endblock E in K with $p(x_E) \in L$ and a directed edge from $L \in B$ to $K \in A'$ for each endblock E in L with $p(x_E) \in K$. Note that every limb other than $\{a, a'\}$ still has an outneighbour in H .

For this H Lemma 4.1 and Lemma 4.2 still hold with the same proofs as before. These two ensure that H still contains a connected component C consisting entirely of limbs and not containing the limb $\{a, a'\}$. Indeed, since b has at least two limbs, pick a limb $L \in B$ not containing E_v and let C be the connected component of H with $L \in C$. As v is the unique neighbour of a in B and $v \notin L$, if $\{a, a'\} \in C$ then H would contain a path of length three, contradicting Lemma 4.1. Furthermore, since $\text{Core}(a') \notin A'$, if C did not consist entirely of limbs of a' and b , $\text{Body}(b) \neq \{b\}$ and C contains two vertices of B , contradicting Lemma 4.2. Following the argument as in Section 5, an application of Lemma 5.2 to G_C finishes the proof of Theorem 2.3. \square

7 A Tight Bound

Theorem 2.1 is proved in the exact same manner as Theorem 2.3 but requires a more care and attention to detail in various arguments. We are now ready for its proof.

Proof of Theorem 2.1. The proof is again by induction on d . The base case $d = 2$ is immediate unless a and b are at Hamming distance 2 apart. If this is the case and G is not isomorphic to Q_2 pick any vertex v of G not in the unique 2-cube containing a and b . By 2-connectivity G contains vertex disjoint

$a - v$ and $v - b$ paths, which when combined give a path of length at least 3, as required. Suppose for contradiction that Theorem 2.1 fails for some minimal d and that G is the smallest counterexample.

We now explain how the proofs above in Sections 3-6 can be altered to give this stronger result. To begin we again assume there is some splitting direction i of G such that forming G_a and G_b as before, $d_{G_a}(a) \geq 2$ and $d_{G_b}(b) \geq 2$. The first lemma of this section says that Lemma 3.4 still applies to G_a and G_b .

Lemma 7.1.

- (i) *Every endblock of G_a which does not contain a in its interior must contain at least two exit vertices.*
- (ii) *G_a is not 2-connected.*
- (iii) *a does not lie in the interior of an endblock in G_a .*
- (iv) *a must have at least two limbs.*

Proof. (i) The proof in this case needs only a slight variation, as if we apply our strengthened induction hypothesis to E as Lemma 3.4(i), we are guaranteed to find an $a - x$ path of length at least $2^d - 2$ and continuing this path as before we adjoin at least one more vertex x with edge $xp(x)$ giving a bound of at least $2^d - 1$, as required.

(ii) The modification for the proof of this case is a little more demanding. Suppose for contradiction that G_a is 2-connected.

First suppose G_b is not 2-connected. If there exists an endblock E in G_b such that $b \notin E$, we have a path in G_b of length at least 1 from b to $\text{cutv}(E)$. We claim E contains a $\text{cutv}(E) - x$ path of length at least $2^{d-1} - 1$ where x is some exit vertex of E with $p(x) \neq a$. If E is isomorphic to the $(d-1)$ -dimensional cube Q_{d-1} every interior vertex of E must be an exit vertex of E . Therefore we can take any $x \in \text{int}(E)$ with $p(x) \neq a$ at odd distance from $\text{cutv}(E)$. If E is not isomorphic to the $(d-1)$ -dimensional cube Q_{d-1} , by (i) E must contain an exit vertex x with $p(x) \neq a$ and by induction on Theorem 2.1 E contains a path of length $2^{d-1} - 1$ from $\text{cutv}(E)$ to x .

Combining the appropriate one of these two with the $a - p(x)$ path of length $2^{d-1} - 2$ in G_a guaranteed by induction on Theorem 2.1, G contains an $a - b$ path of length at least $1 + (2^{d-1} - 1) + 1 + (2^{d-1} - 2) = 2^d - 1$ a contradiction. So b must lie in *every* endblock E_1, \dots, E_t of G_b . Note that since $t \geq 2$ this implies $b \notin \text{int}(E_i)$ for any i .

As with E above, E_1 must have an exit vertex x such that E_1 contains a $x - b$ path of length at least $2^{d-1} - 1$, with $p(x) \neq a$. If G_a were not isomorphic to Q_{d-1} , it contains a path of length $2^{d-1} - 1$ from a to $p(x)$. Combining these two with the edge $xp(x)$ we obtain an $a - b$ path of length $2^d - 1$. Therefore we can assume G_a is isomorphic to Q_{d-1} . For $t \geq 2$ none of the E_1, \dots, E_t can be isomorphic to Q_{d-1} as G_a would have to receive too many edges. Now since G_a is isomorphic to the cube, some endblock E_i must contain an exit vertex x with $p(x) \neq a$ such that x is at even Hamming distance from b . But then since E_i is not isomorphic to Q_{d-1} , by induction E_i contains an $x - b$ path of length at least 2^{d-1} . By induction on Theorem 2.1 applied to G_a , it contains an $a - p(x)$ path of length at least $2^{d-1} - 2$. Combining these paths via the edge $xp(x)$ gives a desired path from a to b of length $2^d - 1$.

The case when G_b is 2-connected is very similar. We can obtain two paths of length at least $2^{d-1} - 1$ in G_a and $G_b = E_1$ if neither of the two are isomorphic

to Q_{d-1} and if one is isomorphic to Q_{d-1} we use the same argument as in the case where G_a is isomorphic to Q_{d-1} and $t \geq 2$ above.

(iii) This is similar to (ii) but a little easier. Again, taking the endblocks E and E' as in the proof of Lemma 3.4(iii), we can find a $a - \text{cutv}(E)$ path P_1 in E of length at least $2^{d-1} - 2$ and a $\text{cutv}(B) - x$ path P_3 in E' of length at least $2^{d-1} - 1$ where x is an exit vertex of E' . Now if $p(x) = a$ then combining these we may only obtain an $a - b$ path of length $2^d - 2$. Instead, using (i) we can choose the exit vertex x so that $p(x) \neq a$. This gives the extra edge on the path required.

(iv) Again follows from (ii) and (iii). \square

The above modifications demonstrate the main problem moving from the bounds in Theorem 2.3 to Theorem 2.1 – on combining endblock paths together without any care as before, we are usually left short by one vertex. To get around this there are two small tricks, both of which were demonstrated above: we try to ensure that we form a path from a to b with at least two endblock paths, so that one of these paths has length at least $2^{d-1} - 1$ and squeeze in another vertex on the way as in Lemma 7.1(iii) above or using a parity argument as in the case where G_a is isomorphic to Q_{d-1} we may be able to show that a given endblock E which we know is not isomorphic to Q_{d-1} contains an exit vertex at even distance away from its cutvertex. This second approach was used in Lemma 7.1(ii) and allows us to find an endblock path of length 2^{d-1} in E which when combined with any other endblock path would give the required bound.

As neither of these options can be guaranteed given the statement of Proposition 3.5 we should not expect to be able to prove it. Now Proposition 3.5 was important in defining our interaction graph H in Section 4 and 5. Can we find a way to define H not depending on this?

We look towards an slightly altered construction for H . Note that unlike our first construction of H in Section 4, since we cannot in general appeal to Proposition 3.5, we now have the possibility that an exit vertex of an endblock in G_a is joined to an exit vertex of an endblock in G_b . It is necessary to pick these exit vertices x_E such that $x_F \neq p(x_E)$ for every two endblocks E and F in G_a and G_b respectively in order to ensure that we can still find 2-connected subgraphs of G in connected components of H as in Lemma 5.2. Can we always find such x_E and x_F ?

The answer is that we can. For each endblock E of a or b , as was seen in Lemma 7.1(ii), there exists at least two exit vertices x of E such that E contains a path of length at least $2^{d-1} - 1$ from $\text{cutv}(E)$ to x . We choose one of these for x_E with the condition $p(x_E) \neq a, b$, which is always possible. What prevents us from choosing such an x_E and x_F with $x_F = p(x_E)$? Because joining the endblock paths from $\text{cutv}(E)$ to x_E and the path from x_F to $\text{cutv}(F)$ in E and F respectively, with the edge $x_E x_F$ we would obtain a path of length at least $(2^{d-1} - 1) + 1 + (2^{d-1} - 1) = 2^d - 1$ which could be extended to an $a - b$ path. Therefore we can choose our x_E for all endblocks E of G_a and G_b such that $x_E \neq p(x_F)$.

We now take our interaction graph $H = \{A, B, \vec{E}\}$ to be a bipartite multi-graph whose bipartition consists of the limbs of a and b respectively. Again we additionally adjoin $\text{Core}(a)$ and $\text{Core}(b)$ to A and B respectively if they are

non-empty. Add a directed edge to H for every endblock E in G_a from $K \in A$ to $L \in B$ if E is an endblock of limb K with $x_E \in K$ and $p(x_E) \in L$. Similarly add a directed edge to H for every endblock F in G_b from $L \in B$ to $K \in A$ if F is an endblock of limb L with $x_F \in L$ and $p(x_F) \in K$. Note again that every limb in H has outdegree at least 1 as it contains an endblock.

We can actually say a lot more about H in the case where one of a or b is not a cutvertex in G_a or G_b - we can guarantee that $p(x_E) \notin \text{int}(F)$ for all endblocks E and F of G_a and G_b . Indeed, suppose we have $p(x_E) = y \in \text{int}(F)$ for some endblocks E of G_a and F of G_b say in limbs K and L respectively. Then by choice of x_E and induction on Theorem 2.1 E contains a $\text{cutv}(E) - x_E$ path of length at least $2^{d-1} - 1$ and F contains a $y - \text{cutv}(F)$ path of length $2^{d-1} - 2$. This gives a path of length $2^d - 2$ from $\text{cutv}(E)$ to $\text{cutv}(F)$, which extends to a path from $\text{Joint}(K)$ to $\text{Joint}(L)$. We must have that either $\text{Joint}(K) \neq a$ or $\text{Joint}(L) \neq b$ since both are not cutvertices. Therefore, extending this path we obtain a path of length at least $2^d - 1$ from a to b , a contradiction.

The property that $p(x_E) \notin \text{int}(F)$ for all endblocks E and F of G_a and G_b was used crucially throughout the proof of Theorem 2.3, and knowing it in the case where one of a or b is not a cutvertex actually allows the proofs of Lemma 4.1 and 4.2 to proceed identically in these cases. The focus therefore will be on obtaining Lemma 4.1 in the case where both a and b are cutvertices of G_a and G_b respectively - we need not worry about Lemma 4.2 as it does not apply to this case. While Lemma 4.1 again holds true, a little more work is required than in the original proof in Section 4.

Lemma 7.2. *H cannot contain an undirected path of length three.*

Proof. From the above discussion we need only consider the case in which a and b are cutvertices of G_a and G_b respectively. If one of the interior vertices on this path has two out-neighbours in Q the same argument as in the original proof will create a path between two exit vertices in this limb of length at least $2^d - 2$. Extending this path through Q as in Theorem 4.1 gives us an $a - b$ path of length at least $2^d - 1$. Similarly, if both endvertices on this path have out-neighbours in Q (that is $\overrightarrow{V_0V_1}$ and $\overleftarrow{V_2V_3}$ are edges of Q) we can find a paths of length at least $2^{d-1} - 1$ in both V_0 and V_3 , which again can be joined through Q to give an $a - b$ path of length at least $2^d - 1$ as required. This just leaves the case of a directed path

$$\overrightarrow{V_0V_1}, \overrightarrow{V_1V_2} \text{ and } \overrightarrow{V_2V_3}. \quad (1)$$

While here we obtain a path of length at least $2^{d-1} - 1$ from the edge $\overrightarrow{V_0V_1}$ as before, in V_1, V_2, V_3 we might not be able to guarantee a full endblock path. Indeed, we may now have the possibility that the edges $\overrightarrow{V_{i-1}V_i}$ and $\overrightarrow{V_iV_{i+1}}$ correspond to one edge entering an endblock E by a vertex x in its interior and the other edge leaving E by a vertex y in its interior. This does not allow us to use induction on Theorem 2.1 as $\text{cutv}(E) \in E - \{x, y\}$ may have degree lower than $d - 1$.

However, if this does not happen at one of V_1 or V_2 the same proof applies. Also if the exit vertex x of V_2 guaranteed by $\overrightarrow{V_2V_3}$ had $p(x)$ in the interior of an endblock of V_3 we would be able to find our two endblock paths, one in V_0 as mentioned already of length at least $2^{d-1} - 1$ and the other in V_3 of

length at least $2^{d-1} - 2$. Since joining both of these through V_1 and V_2 joins at least four more vertices onto these paths, we can extend them to form an $a - b$ path of length at least $2^d + 1$, more than required. Therefore we must have $p(x) \notin \text{int}(E)$.

But now take any outneighbour of V_3 in H . Combined with our path Q above it is easily seen we can obtain a path Q' of length three which is either (i) not of the form (1), or (ii) contains V_3 as an interior vertex and allows for a full endblock path to be build through it. In both cases we are done. \square

The fact that Theorem 7.2 holds for G again enables us to guarantee that H has at least two connected components. As before Lemma 4.2 again gives a connected component C of H consisting entirely of limbs.

Now Theorem 5.2 still holds with an identical proof – the slight variation in the definition of H allows us to guarantee that $|S|, |T| \geq 2$ in the case where C consist of exactly two vertices and that $t' \geq 1$ when it consists of at least three, which ensures 2-connectivity. This time however, it guarantees a path of length at least $2^{d-1} - 2$ from any $v \in J - \{a', b'\}$ to a' not containing b or to b' not containing a . Take J in G_C as guaranteed by Lemma 5.2.

Lemma 7.3. *There does not exist an edge in G from $v \in J - \{a', b'\}$ to $G - J$*

Proof. For contradiction suppose $v \in J - \{a', b'\}$ had a neighbour w outside of J . Without loss of generality take $v \in G_b$. Then $w \notin G_C$ by Theorem 5.2(i) and so $w \in K$ for some limb K of a say or $w \in \text{Core}(a)$. First take $w \in K$, $K \in C'$ for some connected component C' of H .

If $w \notin \text{int}(E)$ for some endblock E of K take a $w - x_E$ path P_2 in K of length at least $2^{d-1} - 1$, where x_E is an exit vertex of E , which exists by induction on Theorem 2.1. Combining this with the path P_1 given from Lemma 5.2(ii) in J from a' to v of length $2^{d-1} - 2$ and the edge $x_E p(x_E)$ we have a path $P_1 v w P_2 x_E p(x_E)$ from a' to $p(x_E)$ of length at least $(2^{d-1} - 2) + 1 + (2^{d-1} - 1) + 1 = 2^d - 1$. But this path extends to a path from a to b , a contradiction.

If $w \in \text{int}(E)$ for some endblock E of K , we would like to combine the $\text{Joint}(K) - w$ path guaranteed by induction on E with the $v - b'$ path in J as given by Lemma 5.2(ii) using the edge wv . This path extends to an $a - b$ path but may only have length $(2^{d-1} - 2) + 1 + (2^{d-1} - 2) = 2^d - 3$, too little for us.

Instead, look at an outneighbour of K in H . If this is a limb then C' must consist entirely of limbs (by Lemma 4.2) and therefore contains a path of the form \overleftarrow{KW} or of the form KVW where \overleftarrow{VW} is an edge of H as all limbs have at least one outneighbour in H . This allows us to build a path P from w to $\text{Joint}(W)$ of length at least $2^{d-1} + 1$ in C' . Now combining P with an appropriate path from Lemma 5.2(ii) via the edge vw we obtain a path that extends to an $a - b$ path of length at least $(2^{d-1} - 2) + 1 + (2^{d-1} + 1) = 2^d$, as required – take this path to be the $a' - v$ path if $W \in B$ or the $b' - v$ path if $W \in A$.

If the outneighbour of K in H is $\text{Core}(b)$, again using Lemma 5.2(ii) we can find a $b' - v$ path in J of length at least $2^{d-1} - 2$ which extends through C' to give a $y - b'$ path P_2 of length at least 2^{d-1} , where $y \in \text{Body}(b)$. Now H must have a third connected component C'' containing a limb of b since b has at least two limbs and only one element of B can lie in a component by Lemma 4.2. This component gives an $a - z$ path P_1 of length at least $2^{d-1} - 2$ where again $z \in \text{Body}(b)$ and P_1 and P_2 are disjoint. As in the proof of Lemma 4.1 we can join P_1 and P_2 together in $\text{Core}(a)$ with a small use of 2-connectivity to give an

$a - b$ path of length at least $2^d - 1$ as required. This completes the case when $w \in K$.

Finally, the case where $w \in \text{Core}(a)$ follows a similar argument. \square

Using Lemma 7.3 we can apply Theorem 2.1 to J taking a' in place of a and b' in place of b . This guarantees a path of length $2^d - 2$ from a' to b' . Moreover, unless J is isomorphic to Q_d with $a = a'$ and $b = b'$ where a and b are at even Hamming distance, J contains a path of length at least $2^d - 1$ between a and b , so we may assume this is the case. Since G is not isomorphic to Q_d , the graph $G' = G[V(G) - J \cup \{a, b\}]$ is non-empty and all v in $G' - \{a, b\}$ have degree at least d in G' .

If a and b both have more than two limbs in G' , they guarantee that G' is 2-connected. In this case since $|G'| < |G|$ we can apply Theorem 2.1 to G' . This gives an $a - b$ path in G' of length at least $2^d - 1$ unless G' is isomorphic to Q_d . If G' was isomorphic to Q_d then J and G' would both contain the subcube containing a and b , which contains at least four points since a and b are at even Hamming distance. But from construction G' and J only share a and b , so G' is not isomorphic to Q_d and therefore contains an $a - b$ path of length at least $2^d - 1$.

If one of a and b has exactly one limb, a say, G' must be of the form G_C for some component C of H (as all limbs of b must have a out-neighbour in H). But then we can apply Theorem 5.2 to G' to obtain a 2-connected subgraph \tilde{J} and vertices $\tilde{a} \in G'_a$ and $\tilde{b} \in G'_b$. As in Lemma 5.2(i) for any $v \in \tilde{J} - \{\tilde{a}, \tilde{b}\}$, \tilde{J} contains all neighbours of v in $G_C = G'$. As such v can have no neighbours in G other than those in G' we have $d_{\tilde{J}}(v) = d_G(v) \geq d$. Theorem 2.1 now holds for \tilde{J} taking \tilde{a} and \tilde{b} in place of a and b so \tilde{J} contains a $\tilde{a} - \tilde{b}$ path of length at least $2^d - 2$ which extends to an $a - b$ path in G' of length at least $2^d - 2$ in G . Again as above, since J and \tilde{J} cannot both be isomorphic to Q_d if a and b are at even Hamming distance, \tilde{J} must contain an $a - b$ path of length at least $2^d - 1$, as required.

Lastly, we show that the degree condition can again be removed. As in Section 6 we can assume that $d_{G_b}(b) \geq 2$ and that a has a neighbour $v \in G_b$, $v \neq b$ with $v \in E_v$ an endblock of G_b – the proof of Lemma 6.1 is identical. In order for the previous argument to be inapplicable a must have exactly one neighbour a' in G_a . Again we switch attention to the limbs of a' and b to construct the interaction digraph H . Note that both a' and b have at least 2 limbs.

Now each endblock $F \neq E_v, \{a, a'\}$ of a' or b has at least two exit vertices x and y such that F contains a path of length at least $2^{d-1} - 1$ from $\text{cutv}(F)$ to x and to y . We choose one of these for x_F with the condition that $p(x_F) \notin \{a, a', b\}$, which is possible since $p(a) \in E_v$. Additionally, let $x_{E_v} = v$. Using these vertices construct the interaction digraph H as before.

We claim that again $p(x_E) \notin \text{int}(F)$ for all endblocks E and F of a' and b , when $F \neq \{a, a'\}$. Otherwise there exists a path P_1 from a' to x_E of length at least $2^{d-1} - 1$ and a path P_2 from $p(x_E)$ to b of length at least $2^{d-1} - 2$. But then we have a path $aa'P_1x_Ep(x_E)P_2$ of length at least $1 + (2^{d-1} - 1) + 1 + (2^{d-1} - 2) = 2^d - 1$ from a to b , a contradiction.

This allows us to again establish Lemma 4.1 and Lemma 4.2 for G since the problem of entering and leaving an endblock E through vertices in its interior

cannot happen. These give a component C of H consisting entirely of limbs of a' and b , with $\{a, a'\} \notin C$. Now we can apply Lemma 5.2 to G_C to obtain J , a'' , b'' as before.

If no $u \in J - \{a'', b''\}$ has a neighbour outside of J then by induction on Theorem 2.1 J contains an $a'' - b''$ path P of length at least $2^d - 2$. As extending P to an $a - b$ path adds at least one more edge aa' , G contains an $a - b$ path of length at least $2^d - 1$, a contradiction. Therefore we may assume some $u \in J - \{a'', b''\}$ has a neighbour w outside of J .

Now notice that the proof of Lemma 5.2(ii) actually gives an $a'' - u$ path not containing b and a $b'' - u$ path not containing a in J of length at least 2^{d-1} in this case. Indeed if we are looking for such an $a'' - u$ path, as before J contains an endblock F on the opposite side of J to u (that is $F \subset J \cap G_a$ if $u \in G_b$ and $F \subset J \cap G_b$ if $u \in G_a$) for which we can find vertex disjoint paths between $\{a'', u\}$ and $\{\text{cutv}(F), y\}$ for some vertex $y \in \text{int}(F)$. The path containing u here must have length at least 2 since $p(u) = w$ which lies outside of J . Combining these with the $\text{cutv}(F) - y$ path of length $2^{d-1} - 2$ in F gives the required path.

If $w \in G_{C'}$ with C' a component of H not containing $\{a, a'\}$ then the remainder of the proof is exactly as in the case where $d_{G_a}(a) \geq 2$. So we can assume $\{a, a'\} \in C'$. Clearly $w \neq a$ as $u \neq a', v$. If $w \in \text{int}(E_v)$ then E_v contains a $\text{cutv}(E_v) - w$ path of length at least $2^{d-1} - 2$. Combining this path via the edge uw with the $a'' - u$ path of length at least 2^{d-1} in J we have an $a'' - \text{cutv}(E_v)$ path of length at least $2^d - 1$ which extends to an $a - b$ path, as required. If $w \notin \text{int}(E_v)$ then $G_{C'}$ contains a $w - a$ path P_1 of length at least $2^{d-1} - 1$ passing through E_v . Combining this with the $u - b'$ path P_2 in J of length at least 2^{d-1} with the edge uw we have an $a - b'$ path $P_1 w u P_2$ of length at least 2^d . As this extends to an $a - b$ path, this finishes the proof of Theorem 2.1. \square

8 Generalizations

The reader might notice that we have used very little about Q_n in the proof of Theorem 2.1. The n -dimensional grid \mathbb{Z}^n is the graph whose vertex set consists of n -tuples with entries in \mathbb{Z} and in which two vertices x and y are adjacent if $|x_i - y_i| = 1$ for some $i \in [n]$ and $x_j = y_j$ for all $j \neq i$. The next theorem extends Theorem 2.1 (and therefore Theorems 1.1 and 1.3) to subgraphs of \mathbb{Z}^d .

Theorem 8.1. *Let G be a 2-connected subgraph of \mathbb{Z}^n and $a, b \in V(G)$. Suppose that $d(z) \geq d$ for all $z \in V(G) - \{a, b\}$. Then a and b are joined by a path of length at least $2^d - 2$. Furthermore unless G is isomorphic to Q_d with a and b at even Hamming distance from each other, G contains an $a - b$ path of length $2^d - 1$.*

Proof. The crucial property of \mathbb{Z}^n here is that we can always find a splitting of G into two connected pieces, G_a and G_b with $a \in G_a$ and $b \in G_b$ such that $d_{G_a}(a) \geq 1$ and $d_{G_b}(b) \geq 1$ and all $v \in G$ lose at most one neighbour in their piece. Indeed, taking some coordinate j on which a and b differ, say with $a_j > b_j$, let G_1 be the induced subgraph of G consisting all vertices v with $v_j \geq a_j$ and G_2 be the induced subgraph of G consisting of all w for which $w_j < a_j$. Again with the same modification to these graphs as in Lemma 2.2 we

obtain connected graphs G_a and G_b with the required degree conditions. From here on the proof is identical to that of Theorem 2.1. \square

Moreover, the same proof also extends to subgraphs of the discrete torus C_k^n provided $k \geq 4$. Now we cannot expect a bound of the form $C2^d$ as above for subgraphs of the discrete torus C_3^d as this graph has minimum degree $2d$ but only 3^d points. This shows that given a subgraph G of C_3^n of minimal degree at least d we cannot in general guarantee a path of length more than $3^{\frac{d}{2}} - 1$ in G .

Why does our approach not work in this case? The main reason is that we cannot guarantee a partition into two subgraphs such that all vertices lose at most one neighbour in their piece. Can we still guarantee an exponentially long path in this case?

The following general result shows that we can.

Theorem 8.2. *Let $k \in \mathbb{N}$ and G be a 2-connected graph with $a, b \in V(G)$. Suppose $d(v) \geq d$ for all $v \in V(G) - \{a, b\}$. Furthermore, suppose that G has the following property:*

Given any two vertices $x, y \in G$, there is a partition of $V(G)$ into two sets X and Y with $x \in X$ and $y \in Y$ such that $d_{G[X]}(v) \geq d(v) - k$ for all $v \in X$ and $d_{G[Y]}(v) \geq d(v) - k$ for all $v \in Y$.

Then G contains an $a - b$ path of length at least $2^{\frac{d}{k+2}}$.

Note that if the property above holds for G , it also holds for all subgraphs of G . As an immediate corollary of Theorem 8.2 we have the following:

Corollary 8.3. *Every subgraph of C_3^n of minimum degree at least d contains a path of length at least $2^{\frac{d}{4}}$.*

It would be interesting to decide what the correct lower bounds for the length of the longest path in subgraphs of C_3^n with minimum degree at least d .

Conjecture 8.4. *Given a subgraph G of C_3^n with minimum degree at least d , G must contain a path of length at least $3^{\frac{d}{2}} - 1$.*

Another consequence of Theorem 8.2 is the following result for product graphs.

Theorem 8.5. *Let G_1, \dots, G_l be graphs with maximum degree at most k . Then given any subgraph G of the Cartesian product graph $\prod_{i=1}^l G_i$ of minimum degree at least d , G contains a path of length at least $2^{\frac{d}{k+2}}$.*

The proof of Theorem 8.2 is similar to that of Theorem 2.1 but shorter.

Proof. The proof is again by induction on d . It suffices to prove the result for $d \geq k+4$ as otherwise it follows from 2-connectivity. As in the proof of Theorem 2.1 we wish to split G into two subgraphs G_a and G_b with $a \in G_a$ and $b \in G_b$, which is the motivation for the above splitting property. However, simply taking a and b in place of x and y might not be useful as both a and b can have degree as low as two in G in which case in the partition guaranteed above a may end up with all its neighbours in Y . Instead we pick a neighbour $a' \neq b$ of a and a neighbour $b' \neq a$ of b . The fact that G is 2-connected ensures it is possible

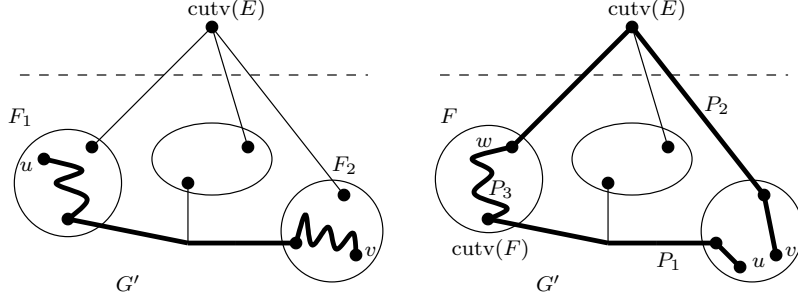


Figure 5: Cases where G' is not 2-connected in Lemma 8.6

to pick $a' \neq b'$. Now take the partition guaranteed from our splitting property above with $x = a'$ and $y = b'$. Moving a to X and b to Y as needed we have that $d_{G[X]}(v) \geq d(v) - k - 1$ for all $v \in X - a$ and similarly for $v \in Y - b$. Both a and b now have at least 1 neighbour in $G[X]$ and $G[Y]$ respectively. Finally, denoting the connected component of $G[X]$ containing a by C_a , let G_b be the connected component of $G - C_a$ containing b and $G_a = G[V(G) - V(G_b)]$. Note that G_a and G_b are connected with $a \in G_a$, $b \in G_b$. Moreover, $d_{G_a}(v) \geq d(v) - k - 1$ for all $v \in G_a - a$ and $d_{G_b}(v) \geq d(v) - k - 1$ for $v \in G_b - b$.

We will again analyse the block-cutvertex decompositions of G_a and G_b . The following lemma will be very useful below.

Lemma 8.6. *Let E be an endblock of G_a or G_b with $a, b \notin \text{int}(E)$. Then given any two vertices $u, v \in \text{int}(E)$, $G[E]$ contains a path of length at least $2^{\frac{d-k-2}{k+2}}$ from u to v .*

Proof. Look at the block-cutvertex decomposition of $G' = G[E] - \text{cutv}(E)$. Since E is 2-connected (as $d \geq k + 4$), G' is connected and $\text{cutv}(E)$ must have a neighbour in the interior of every endblock of G' . Note that every vertex $v \in G'$ has $d_{G'}(v) \geq d_G(v) - k - 2$. In particular, since $d \geq k + 4$ each endblock F of G' is 2-connected and has at least three vertices so that we can by induction apply Theorem 8.2 to it. If G' is 2-connected then by induction on Theorem 8.2 G' contains the desired path from u to v . Thus we may assume that G' is not 2-connected. If $u \in \text{int}(F_1)$ and $v \in \text{int}(F_2)$ where F_1 and F_2 are two distinct endblocks of G' then by induction on Theorem 8.2, $G[F_1]$ and $G[F_2]$ contain $u - \text{cutv}(F_1)$ and $\text{cutv}(F_2) - v$ paths respectively, each of length at least $2^{\frac{d-k-2}{k+2}}$. Joining $\text{cutv}(F_1)$ to $\text{cutv}(F_2)$ by a third path in G' and combining all three of these paths, we get a $u - v$ path of length at least $2^{\frac{d}{k+2}}$, as required. Therefore since G' contains at least two endblocks, we can assume that one of these, say F , does not contain u or v in its interior. Contracting $\text{int}(F)$ down to a single vertex in $G[E]$, the resulting graph is still 2-connected. Therefore, as in the proof of Lemma 5.2, $G[E]$ contains two vertex disjoint paths P_1 and P_2 from the set $\{u, v\}$ to $\{\text{cutv}(F), w\}$ for some $w \in \text{int}(F)$, with $(P_1 \cup P_2) \cap (F - \{\text{cutv}(F), w\}) = \emptyset$. Now using induction on Theorem 8.2 in F , it contains a path P_3 of length $2^{\frac{d-k-2}{k+2}}$ from $\text{cutv}(F)$ to w . Piecing P_1 , P_2 and P_3 together we obtain our desired path. \square

Again we have:

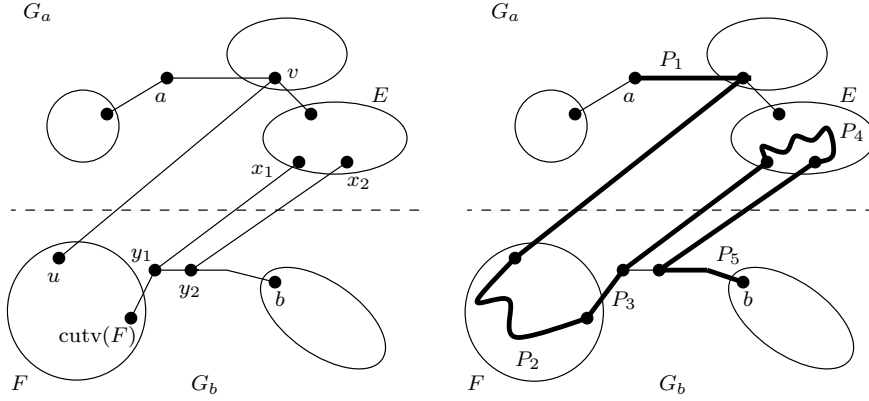


Figure 6: Path created in Theorem 8.2

Proposition 8.7. *Let E be an endblock of G_a not containing a and F an endblock of G_b not containing b . Then G does not contain an edge from $\text{int}(E)$ to $\text{int}(F)$*

Proof. Exactly as in Proposition 3.5. \square

Lemma 8.8. *We have the following:*

- (i) *Given any endblock E of G_a not containing a , there are two disjoint edges from $\text{int}(E)$ to G_b in G .*
- (ii) *G_a contains an endblock not containing a .*

Proof. (i) E must have an exit vertex x_1 , with neighbour $y \in G_b$, as G is 2-connected. If it had only one, $G' = G[E]$ is 2-connected and every $v \in G' - \{x_1, \text{cutv}(E)\}$ has degree at least d . Therefore, by induction on Theorem 8.2, G' contains a path of length at least $2^{\frac{d}{k+2}}$ from x_1 to $\text{cutv}(E)$. Extending this path from $\text{cutv}(E)$ to a in G_a and from y to b in G_b we obtain an $a-b$ path of desired length. Therefore we may assume E contains a second exit vertex x_2 . Now if the vertices in $\text{int}(E)$ were only adjacent to y in G_b , x_1y and x_2y must be edges of G . Then $G'' = G[E \cup \{y\}]$ is 2-connected and $d_{G''}(v) \geq d$ for every $v \in G'' - \{y, \text{cutv}(E)\}$. By Theorem 8.2 G'' contains a $\text{cutv}(E) - y$ path of length at least $2^{\frac{d}{k+2}}$. Again, extending this to a path from a to b , we have an $a-b$ path of length at least $2^{\frac{d}{k+2}}$. Therefore we may assume the two edges exist or we are done.

- (ii) The proof is almost identical to the proof of Lemma 3.4(ii). \square

Take an endblock E of G_a not containing a , as guaranteed by Lemma 8.8(ii). We can choose E such that a and all $v \in G_a - E$ not contained in the interior of an endblock of G_a lie in the same connected component of $G_a - E$ (e.g. pick a block B in G_a containing a and choose E to be a block at maximum distance from B in $\mathcal{B}(G_a)$). Let x_1y_1 and x_2y_2 be the disjoint edges of G with $x_1, x_2 \in \text{int}(E)$ and $y_1, y_2 \in G_b$ guaranteed by Lemma 8.8(i). By Proposition 8.7, $y_1, y_2 \notin \text{int}(F)$ for all endblocks F of G_b not containing b in its interior.

Now looking at the block-cutvertex decomposition of G_b we can choose two vertex disjoint paths in G_b from $\{y_1, y_2\}$ to $\{b, \text{cutv}(F)\}$ where F is some endblock of G_b not containing b . Lets say that these paths are P_3 from $\text{cutv}(F)$ to y_1 and P_5 from y_2 to b . Applying Lemma 8.8(i) to F we see that there exists $u \in \text{int}(F)$ adjacent to some $v \in G_a$, $v \neq \text{cutv}(E)$. Furthermore, by Proposition 8.7 $v \notin \text{int}(E')$ for any endblock E' of G_a . From our choice of E there exists an $a - v$ path P_1 in $G_a - E$. Finally by induction on Theorem 8.2, F contains a $u - \text{cutv}(F)$ path P_2 of length at least $2^{\frac{d-k-1}{k+2}}$ and by Lemma 8.6 E contains an x_1x_2 path P_4 of length at least $2^{\frac{d-k-2}{k+2}}$. Combining these five paths we obtain an $a - b$ path $P = P_1vuP_2P_3y_1x_1P_4x_2y_2P_5$ of length at least $2^{\frac{d-k-1}{k+2}} + 2^{\frac{d-k-2}{k+2}} > 2^{\frac{d}{k+2}}$ as required. \square

The cycle analogues of the above theorems can be obtained in a similar fashion to the proof of Theorem 1.3 from Theorem 2.1.

As mentioned in the Introduction, we do not know the correct bound for the length of the longest path in a subgraph of Q_n when the minimum degree condition in Theorem 1.1 is replaced by an average degree condition. Is the following possible?

Conjecture 8.9. *Every subgraph of Q_n with average degree at least d contains a path of length at least $2^d - 1$.*

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